

# MOMENTS APPROACH TO THE CONCENTRATION PROPERTIES OF TRUNCATED VARIATION

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**ABSTRACT.** In this paper we show an alternative approach to the concentration of truncated variation for stochastic processes on a real line. Our method is based on the moments control and can be used to generalize the results to the case of processes with heavy tails.

## 1. INTRODUCTION

Let  $X = X(t)$ ,  $t \in [0, 1]$  be a real valued stochastic process with càdlàg trajectories. In this paper we are interested in concentration properties of the truncated variation of  $X$ . It is well-known that usually the total variation for semimartingales is infinite, whereas it is always possible to define for a given  $c > 0$

$$(1.1) \quad TV^c(X) := \sup_n \sup_{0=t_0 < t_1 < \dots < t_n=1} \sum_{i=1}^n \max\{|X(t_i) - X(t_{i-1})| - c, 0\}.$$

It was proved in [13] that for any continuous semimartingale  $X$  on  $[0, 1]$  we have

$$(1.2) \quad c \cdot TV^c(X) \rightarrow_{c \downarrow 0} \langle X \rangle_1 - \langle X \rangle_0 \text{ a.s.},$$

where  $\langle \cdot \rangle$  denotes the quadratic variation of  $X$ . In this way truncated variation may be useful in the so called pathwise approach to stochastic integration.

In the paper [4] we have proved concentration inequalities for various processes whose increments decay exponentially fast. In this paper we introduce a new approach based on the method of moments. In this way it is possible to establish bounds on moments of the truncated variation under much weaker assumptions. In particular it should be possible to study some processes with heavy tailed increments.

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## 2. THE MOMENTS CONTROL

We assume that for a given  $k \geq 1$  and all  $s, t \in [0, 1]$

$$(2.1) \quad \|X(t) - X(s)\|_k \leq C_1 k^p |s - t|^q,$$

where  $0 < q < 1$ ,  $p > 0$ . The second condition we require is that if  $d \geq C_2 k^p |s - t|^q$  for some  $s, t \in [0, 1]$  then also

$$(2.2) \quad \|(|X(t) - X(s)| - d)_+\|_k \leq C_2 k^p |s - t|^q f\left(\frac{d}{C_2 k^p |s - t|^q}\right),$$

where the function  $f$  is positive, decreasing and satisfies the following growth condition: for some integer  $r \geq 2^{1/q}$  and any given constant  $C_3$  the following must holds

$$(2.3) \quad \sum_{l=0}^{\infty} r^{l(1-q)} f(C_3 (r^q/2)^{l/p}) < \infty.$$

It means that the function  $f$  must decay slightly faster than the function  $1/x^\alpha$ , where  $\alpha > \frac{p}{q} - p$ . The natural setting in which these condition works is when (2.1) is satisfied for all  $k \geq 1$  and  $p \leq 1$ . Then obviously

$$\mathbf{P}(|X(t) - X(s)| > x e C_1 |s - t|^q) \leq \exp(-x^{\frac{1}{p}}) \text{ for } x \geq 1$$

and thus for all  $x > e C_1 |s - t|^q$

$$\mathbf{P}(|X(t) - X(s)| > x) \leq \exp\left(-\left[\frac{x}{e C_1 |s - t|^q}\right]^{1/p}\right).$$

Thus if  $d > e C_1 |s - t|^q$  then

$$\begin{aligned} \mathbf{E}(|X(t) - X(s)| - d)_+^k &= \int_0^\infty k x^{k-1} \mathbf{P}(|X(t) - X(s)| > x + d) dx \\ &= \int_0^\infty k x^{k-1} \exp\left(-\left[\frac{d+x}{e C_1 |s - t|^q}\right]^{1/p}\right) dx. \end{aligned}$$

Using that  $p \leq 1$  and  $x > 0$  we obtain

$$\left[\frac{d+x}{e C_1 |s - t|^q}\right]^{1/p} \geq \left[\frac{d}{e C_1 |s - t|^q}\right]^{1/p} + \left[\frac{x}{e C_1 |s - t|^q}\right]^{1/p}$$

and hence

$$\begin{aligned} &\int_0^\infty k x^{k-1} \exp\left(-\left[\frac{d+x}{e C_1 |s - t|^q}\right]^{1/p}\right) dx \\ &\leq \exp\left(-\left[\frac{d}{e C_1 |s - t|^q}\right]^{1/p}\right) (e C_1 |s - t|^q)^k \int_0^\infty k p x^{kp-1} \exp(-x) dx \\ &= \exp\left(-\left[\frac{d}{e C_1 |s - t|^q}\right]^{1/p}\right) (e C_1 |s - t|^q)^k \Gamma(kp). \end{aligned}$$

Since  $[\Gamma(kp)]^{\frac{1}{k}} \leq (kp)^p$  we finally get

$$\|(X(t) - X(s)) - d\|_k \leq (kp)^p (eC_1 |s - t|^q) \exp\left(-\frac{1}{k} \left[ \frac{d}{eC_1 |s - t|^q} \right]^{\frac{1}{p}}\right).$$

Therefore (2.2) holds with  $C_2 = eC_1 p^p$  and We use (2.1) and (2.2) to control moments of the truncated variation.

**Theorem 1.** *Suppose that (2.1) and (2.2) are satisfied. Then there exists a universal constant  $K$  which may depend on  $p, q$  such that*

$$\|TV^c(X, T)\|_k \leq K c^{1-\frac{1}{q}} k^{\frac{p}{q}}.$$

**Proof.** Consider given partition  $\Pi_d = \{t_0, t_1, \dots, t_d\}$ , where  $0 \leq t_0 < t_1 < \dots < t_d \leq 1$ . We have to provide a universal upper bound for

$$\sum_{i=1}^d (|X(t_i) - X(t_{i-1})| - c)_+.$$

In order to find such an estimate following paper [4] we introduce the approximation sequence  $(T_n)_{n=0}^\infty$ ,  $T_n \subset T$ . Fix integer  $r > 1$  and let  $T_n = \{kr^{-n} : k = 0, 1, \dots, r^n\}$ . Obviously  $T_n \subset T_{n+1}$  and  $|T_n| = r^n + 1$ . We define neighbourhood of a given point  $t \in T_{n+1} \supset T_n$ , namely

$$I_{n+1}(t) = \{s \in T_{n+1} : |s - t| < 2r^{-n}\}.$$

Clearly  $|I_{n+1}(t)| \leq 2r - 1$ . For a given set  $T_n$  and  $t \in T$ , by  $\pi_n(t)$  we denote the unique point  $s \in T_n$  such that  $s \leq t$  and  $|t - s| < r^{-n}$ . This way we define the function  $\pi_n : T \rightarrow T_n$ . We have  $|t - \pi_n(t)| < r^{-n}$  for all  $t \in T$  and  $\pi_n(s) \leq \pi_n(t)$  if  $s \leq t$ . Note that  $s, t \in T_n$ ,  $s \neq t$ ,  $|s - t| \geq r^{-n}$ . We use the above construction to approximate intervals  $[t_{i-1}, t_i]$  for any consecutive  $t_{i-1}, t_i \in \Pi_d$  where  $i \in \{1, 2, \dots, d\}$ . We denote by  $t_i^n = \pi_n(t_i)$ , it is crucial to observe that for a fixed  $t_i^n$  there cannot be too many candidates for  $t_i^{n+1}$ . Clearly

$$|t_i^n - t_i^{n+1}| \leq |t_i - t_i^n| + |t_i - t_i^{n+1}| < r^{-n} + r^{-n-1} < 2r^{-n}.$$

Consequently  $t_i^{n+1} \in I_{n+1}(t_i^n)$ . Moreover  $t_0^{n+1} \leq t_1^{n+1} \leq \dots \leq t_d^{n+1}$ . Obviously  $(t_i^n)_{n=0}^\infty$  is a path that approximates point  $t_i$  in the sense that  $\lim_{n \rightarrow \infty} |t_i - t_i^n| = 0$ . Since we have to consider all intervals  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, d$  we first classify their lengths. To this aim we define for  $m = 0, 1, \dots$

$$J_m = \{i \in \{1, \dots, d\} : r^{-m-1} < |t_{i-1} - t_i| \leq r^{-m}\}.$$

The approximation paths for the interval  $[t_{i-1}, t_i]$  consists of  $(t_i^n)_{n=m+1}^\infty$  and  $(t_{i-1}^n)_{n=m+1}^\infty$ . We call any pair  $(t_i^n, t_{i-1}^{n+1})$ ,  $n \geq m+1$  a step of the approximation. Observe that

$$|t_i^{m+1} - t_{i-1}^{m+1}| \leq |t_i - t_i^{m+1}| + |t_i - t_{i-1}| + |t_{i-1} - t_{i-1}^{m+1}| < 2r^{-m-1} + r^{-m} \leq 2r^{-m}.$$

Therefore  $t_{i-1}^{m+1} \in I_{m+1}(t_i^{m+1})$  and  $t_i^{m+1} \in I_{m+1}(t_{i-1}^m)$ . We have to prove that for fixed  $u \in T_n$ ,  $v \in I_{n+1}(u) \subset T_{n+1}$  there are at most two different

intervals  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, d$  such that  $t_i^n = u$ ,  $t_i^{n+1} = v$  or  $t_{i-1}^n = u$ ,  $t_{i-1}^{n+1} = v$ .

**Lemma 1.** *Consider  $u \in T_n$ , and  $v \in I_{n+1}(u) \subset T_{n+1}$ . The step  $(u, v)$  may occur in the approximation of  $\Pi_d$  in two ways: either there exists no more than one  $i \in \{1, \dots, d\}$  such that  $i \in J_m$ ,  $m+1 \leq n$  and  $t_{i-1}^n = u \in T_n$ ,  $t_{i-1}^{n+1} = v$  or there exists no more than one  $i \in \{1, 2, \dots, d\}$  such that  $i \in J_{m'}$ ,  $m'+1 \leq n$  and  $t_i^n = u$ ,  $t_i^{n+1} = v$ .*

**Proof.** Recall that  $r \geq 2$ . It suffices to prove that for a given  $i \in J_m$ ,  $n \geq m+1$  points  $t_i^{n+1}$  and  $t_{i-1}^{n+1}$  are different. Indeed since  $t_0^{n+1} \leq t_1^{n+1} \leq \dots \leq t_d^{n+1}$  the property implies that there can be at most one  $i \in \{0, 1, \dots, d\}$  such that  $t_i^{n+1} = v$ . To prove the assertion we use  $|t_i - t_{i-1}| > r^{-m-1}$  which implies that for  $l \geq m+1$

$$\begin{aligned} |t_i^{n+1} - t_{i-1}^{n+1}| &\geq r^{-m-1} - |t_{i-1}^{n+1} - t_{i-1}| - |t_i^{n+1} - t_i| \\ &> r^{-m-1} - 2r^{-n-1} \geq 0 \end{aligned}$$

since  $r \geq 2$ . ■

Let  $m_k$  be defined as  $k^p r^{-(m_k+1)q} < c/M_0 \leq k^p r^{-m_k q}$ , where  $M_0 = 2C_2 e r^{2q}$ . We are ready to state the right upper bound on the truncated variation.

**Lemma 2.** *The following estimate holds*

$$\begin{aligned} \sum_{i=1}^d (|X(t_i) - X(t_{i-1})| - c)_+ &\leq 2 \sum_{n=0}^{m_k} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} |X(u) - X(v)| \\ &+ 2 \sum_{n=m_k+1}^{\infty} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} (|X(u) - X(v)| - 2^{m_k-n-1}c)_+ \end{aligned}$$

**Proof.** First consider given interval  $[t_{i-1}, t_i]$  and  $i \in J_m$ . If  $m \leq m_k$

$$\begin{aligned} (|X(t_i) - X(t_{i-1})| - c)_+ &\leq |X(t_i^{m+1}) - X(t_{i-1}^{m+1})| + \\ &+ \sum_{s \in \{i-1, i\}} \sum_{n=m+1}^{m_k} |X(t_s^n) - X(t_s^{n+1})| \\ &+ \sum_{n=m_k+1}^{\infty} (|X(t_s^n) - X(t_s^{n+1})| - 2^{m_k-n-1}c)_+. \end{aligned}$$

On the other hand if  $i \in J_m$ ,  $m > m_k$  then

$$\begin{aligned} (|X(t_i) - X(t_{i-1})| - c)_+ &\leq (|X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - 2^{m_k-m-2}c)_+ \\ &+ \sum_{s \in \{i-1, i\}} \sum_{n=m+1}^{\infty} (|X(t_s^n) - X(t_s^{n+1})| - 2^{m_k-n-1}c)_+. \end{aligned}$$

It suffices to apply Lemma 1 to finish the proof. ■

Consequently we can bound

$$\begin{aligned} \|TV^c(X, T)\|_k &\leq 2 \sum_{n=0}^{m_k} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} \|X(u) - X(v)\|_k \\ &+ 2 \sum_{n=m_k+1}^{\infty} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} \|(|X(u) - X(v)| - 2^{m_k-n-1}c)_+\|_k. \end{aligned}$$

By the assumption (2.1) we have

$$\|X(u) - X(v)\|_k \leq C_1 k^p |u - v|^q$$

and hence

$$\begin{aligned} 2 \sum_{n=0}^{m_k} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} \|X(u) - X(v)\|_k &\leq 2C_1 \sum_{n=0}^{m_k} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} k^p |u - v|^q \leq \\ &\leq 2C_1 \sum_{n=0}^{m_k} (r^{n+1} + 1)(2r - 1)k^p (2r^{-n})^q \\ &\leq 8C_1 r^2 k^p \sum_{n=0}^{m_k} r^{n(1-q)}. \end{aligned}$$

Using that  $q < 1$  and  $r^{m_k(1-q)} \leq M_0^{\frac{1-q}{q}} k^{\frac{p}{q}-p} c^{-\frac{1-q}{q}}$  we establish the final bound for this part

$$\begin{aligned} 2 \sum_{n=0}^{m_k} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} \|X(u) - X(v)\|_k \\ \leq 8Cr^2 \frac{k^p (r^{(m_k+1)(1-q)} - 1)}{r^{1-q} - 1} \leq D_1(r, q) k^{\frac{p}{q}} c^{1-\frac{1}{q}}, \end{aligned}$$

where  $D_1(r, q) = (8Cr^{3-q}M_0^{\frac{1}{q}-1})/(r^{1-q} - 1)$ . Now we have to bound

$$\|(|X(u) - X(v)| - 2^{m_k-n-1}c)_+\|_k.$$

For this part we need the second assumption (2.2). In order to use the inequality we need that  $d = 2^{m_k-n-1}c \geq C_2 k^p |u - v|^q$ . Note that if  $u \in T_{n+1}$ ,  $v \in I_{n+1}(u)$ ,  $n > m_k$  then  $|u - v| \leq 2r^{-n}$  and hence

$$\begin{aligned} C_2 k^p |u - v|^q &\leq 2^q C_2 k^p r^{-nq} \\ &\leq 2^q C_2 r^{2q} k^p r^{-(n+1-m_k)q} r^{-(m_k+1)q} \leq 2^{-(n+1-m_k)q} c \end{aligned}$$

since  $r \geq 2$  and  $r^{-(m_k+1)q} < c/M_0$  and  $M_0 = 2^q C_2 r^{2q}$ . Therefore by (2.2)

$$\|(|X(u) - X(v)| - d)_+\|_k \leq C_2 k^p |u - v|^q f\left(\frac{d}{C_2 k^p |u - v|^q}\right).$$

Note that for  $u \in T_{n+1}$  and  $v \in I_{n+1}(u)$  due to  $d = 2^{m_k-n-1}$  and  $k^p c^{-1} \leq M_0^{-1} r^{(m_k+1)q}$  we get

$$\begin{aligned} \frac{k^p |u - v|^q}{d} &\leq 2^{n+1-m_k} (2r^{-n})^q k^p c^{-1} \leq \\ &\leq M_0^{-1} (2r^{-n})^q (2^{n+1-m_k} r^{q(m_k+1)}) \leq 4 \cdot 2^q M_0^{-1} (r^q/2)^{-(n-m_k-1)}. \end{aligned}$$

Therefore

$$(2.4) \quad \|(|X(u) - X(v)| - 2^{m_k-n-1}c)_+\|_k$$

$$(2.5) \quad \leq C_2 k^p r^{-qn} f(D_2(r, q)(r^q/2)^{-(n-m_k-1)/p}),$$

where  $D_2(r, q) = (4 \cdot 2^q)^{-1} M_0$ . It remains to sum up all the bounds (2.5)

$$\begin{aligned} &2 \sum_{n=m_k+1}^{\infty} \sum_{u \in T_{n+1}} \sum_{v \in I_{n+1}(u)} (|X(u) - X(v)| - 2^{m_k-n-1}c)_+ \\ &\leq 2C_2 k^p \sum_{n=m_k+1}^{\infty} (r^{n+1} + 1)(2r - 1) r^{-qn} f(D_2(r, q)(r^q/2)^{(n-m_k-1)/p}) \\ &\leq 4C_2 r^2 k^p \sum_{n=m_k+1}^{\infty} r^{n(1-q)} f(D_2(r, q)(r^q/2)^{(n-m_k-1)/p}) \\ &\leq D_3(r, q) k^{\frac{p}{q}} c^{1-\frac{1}{q}} \sum_{n=m_k+1}^{\infty} r^{(n-m_k-1)(1-q)} f(D_2(r, q)(r^q/2)^{(n-m_k-1)/p}), \end{aligned}$$

where  $D_3(r, q) = 4C_2 r^{3-q} M_0^{\frac{1}{q}-1}$ . Note that in the last line we have used that  $c/M_0 \leq k^p r^{-m_k q}$  which implies that

$$k^p \leq M_0^{\frac{1}{q}-1} k^{\frac{p}{q}} c^{1-\frac{1}{q}} r^{-m_k(1-q)}.$$

We have to consider

$$D_4(r, q, p) = D_3(r, q) \sum_{l=0}^{\infty} r^{l(1-q)} f(D_2(r, q, p)(r^q/2)^{l/p})$$

but by our growth condition (2.3)  $D_4(r, q, p)$  is finite and does not depend on  $k$  nor  $m_0$ . Therefore we finally derive

$$\|TV^c(X)\|_p \leq (D_1(r, q) + D_2(r, q) D_4(r, q, p)) k^{\frac{p}{q}} c^{1-\frac{1}{q}}.$$

It ends the proof. ■

The consequence of the above theorem and our argument from the beginning of Section 2 is that if a process  $X$  satisfies (2.1) for all  $k \geq 1$  and  $p \leq 1$  then it also satisfies (2.2) and hence by Theorem 1 its truncated variation satisfies the following concentration inequality.

**Corollary 1.** *Suppose that (2.1) is satisfied for all  $k \geq 1$  and  $p \leq 1$  then*

$$\mathbf{P}(TV^c(X) \geq Duc^{1-\frac{1}{q}}) \leq \exp(-u^{\frac{q}{p}}), \text{ for } u \geq 1,$$

where  $D$  is a universal constant.

In particular Corollary 1 works for any fractional Brownian motions  $X_H$  with Hurst coefficient  $H \in (0, 1)$ . Indeed for each  $k \geq 1$  the process  $X_H$  satisfies

$$\|X_H(t) - X_H(s)\|_k \leq Ck^{1/2}|t - s|^{H/2},$$

so (2.1) holds for  $p = \frac{H}{2}$ . Therefore by Corollary 1 we get

**Theorem 2.** *For a fractional Brownian motion  $X_H$  with Hurst coefficient  $H \in (0, 1)$*

$$\mathbf{P}(TV^c(X_H) > Duc^{1-\frac{1}{H}}) \leq e^{-u^{2H}}, \text{ for } u \geq 1,$$

where  $D$  is a universal constant.

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